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Extended Geroch–Held–Penrose Formalism

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An extension is presented of the well-known Geroch-Held-Penrose (GHP) formalism, itself an extension of the still better known Newman-Penrose (NP) formalism. The extended formalism given here uses only quantities that transform properly under all diagonal transformations of the spin frame, that is, not only under boost-rotations, but also under conformal rescalings. Full use is made of the formalism's symmetry under all discrete operations, that is, under conjugation, the prime operation, and the (modified) Sachs transformations. Just as the GHP formalism is considerably simpler than the NP formalism in the case where a spacelike surface is singled out in a natural way, so the present formalism leads to further simplification when a conformal spacelike surface can be singled out. This is the case, for example, in considerations of future null infinity. In general situations all three formalisms are on an equal footing.

1. INTRODUCTION

The Newman-Penrose (NP) formalism (Newman and Penrose, 1962), also called the spin-coefficient formalism, is well known and has long shown its usefulness, for instance, in finding solutions to the Einstein field equations. Slightly less well known is an extension called the compacted spin-coefficient formalism or Geroch-Held-Penrose (GHP) formalism (Geroch *et al.*, 1973; Penrose and Rindler, 1984). It is on an equal footing with the former, in the sense that either formalism may be used in any one situation, but it leads to considerable simplifications in cases where a spacelike surface (and hence two null directions, l and n) may be singled out in a natural way. It deals only with quantities that "transform properly" under those Lorentz transformations that leave invariant the two null directions, i.e., under boosts in the l–n plane and under rotations in the m–$\bar{m}$ plane perpendicular to these two null directions.

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The formalism to be described in this paper goes one step further and uses only quantities that transform properly under *all* diagonal transformations of the underlying spin frame, that is, not only under boost-rotations, but also under conformal rescalings. It is particularly suited to situations where a conformal spacelike surface can be singled out in a natural way, as happens, for instance, in considerations of future null infinity. However, it must be stressed that it is as general a formalism as the NP or GHP formalisms.

Such a formalism was alluded to by Penrose and Rindler (1984), but was left in a rather rudimentary state. We shall make the following contributions to this problem.

First, Penrose and Rindler seem to consider boost-rotations and conformal rescalings as unrelated entities. We prefer to see them as different diagonal transformations (Ludwig, 1986a) of the spin frame. The ambiguity as to precisely what conformal rescaling to consider now largely disappears.

Second, the differential operators used by Penrose and Rindler turn out, contrary to first appearance, to be more cumbersome than a second, more symmetrical set also briefly mentioned by them.

Third, we shall show in detail what happens to all quantities of interest, such as the spin coefficients and components of the Ricci tensor. We shall also write out all NP equations, i.e., spin-coefficient equations, commutator equations, and Bianchi identities, in the new formalism.

Fourth, with very little extra work the formalism is extended to the complex case. The study of complex general relativity has recently gained some importance (McIntosh and Hickman, 1985) and the NP formalism was found to be readily adaptable to this complex case. The spin frame consists of two dyads (o^A, ι^A) and $(\tilde{o}^A, \tilde{\iota}^A)$, which are complex conjugates of each other in the "real" case, but are independent of each other in the more general complex case. In general, they transform according to independent (instead of complex conjugate) complex 2×2 matrices.

Fifth, in the NP formalism conjugation plays a major role, as do both conjugation and the prime operation in the GHP formalism. A third discrete transformation of the spin frame, the Sachs (star) operation, was briefly mentioned in the original paper of Geroch *et al.* (1984), but was never fully utilized. In the formalism to be described, conjugation and slightly modified versions (Ludwig, 1986b) of both the prime and star operations will play a fundamental role, not the least of which is to minimize the number of equations that need to be written down explicitly. We therefore start by reviewing these discrete operations in the next section.

In a fashion analogous to the GHP notation, we could make full use of the discrete operations described in Section 2 and write, for example, κ, κ', κ^*, κ'^* instead of κ, $-\nu$, σ, $-\lambda$, respectively. We give full details of this logical extension of the GHP notation in Appendix C. However, in order to avoid notational bedlam in the remainder of the paper, we continue to employ the NP names for the various NP quantities and make frequent use of the transformation formulas of Appendix A to write down the primed, starred, and primed starred versions of various equations.

2. THE DISCRETE TRANSFORMATIONS

A recent paper (Ludwig, 1987) dealt with complex Lorentz transformations and complex rescalings in a unified manner, treating them as elements of $GL(2, C) \otimes \tilde{GL}(2, C)$ acting on the spin frame, where $GL(2, C)$ is the complex general linear group in two dimensions. Our terminology and notation are based on this paper.

Another paper (Ludwig, 1986b) discussed in some detail those elements of this group that simply permute the null tetrad vectors. Included among these are the following *discrete transformations*.

1. The basic reflection (*conjugation*), which interchanges tilded and untilded quantities.

2. The (modified) GHP prime transformation (the *prime operation*)

$$L_1 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

3. The (modified) right Sachs transformation (the *star operation*),

$$L_2 = \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

4. Their product (the prime-star operation), the (modified) left Sachs transformation,

$$L_3 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I \right\}$$

where I is the two-dimensional identity matrix. In addition, there are the products of conjugation with each of the last three operations.

The transformation laws, for various quantities of interest, under the first three of these operations are listed in Appendix A. The prime operation commutes with both the star operation and conjugation. The latter two do not commute, however. Instead, for any quantity η being transformed we have

$$(\tilde{\eta})^* = (\widetilde{\eta'^*})$$

Our modified prime operation differs from the GHP prime operation

$$P=\left\{\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\right\}$$

by a pure spin transformation (Ludwig, 1987) $S=\{-iI, iI\}$, i.e., $L_1=SP$. Therefore, L_1 and P have the same effect on tensor-related quantities such as the spin coefficients. Using the fact that $P^2=\{-I,-I\}$, a pure spin transformation, it is not hard to show that for the GHP prime operation, $\eta''=(-1)^{m-n}\eta$, or equivalently, $\eta''=(-1)^{p+q}\eta$ (for the definitions of the weights m, n, p, q see below), whereas for our modified prime operation, we have $\eta''=\eta$.

Let us now turn our attention to the star operation. Our modified version of a right Sachs transformation has been discussed previously (Ludwig, 1986b). It is the original star operation preceded by the basic right conformal transformation

$$\left\{I,\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right\}$$

The latter multiplied each NP variable by +1 or −1; in particular, it multiplies the metric by −1. The main reason for our modification is that there are no longer any "hard-to-remember" minus signs in the transformation laws for the NP variables, as Appendix A2 shows. Note also that $\eta^{**}=\eta$.

The importance of these discrete transformations for our purposes lies in the fact that they can be used to generate the NP equations from a basic few. In fact, using the numbering of the original NP paper (Newman and Penrose, 1962), the spin-coefficient equations may be split into the following eight sets:

(i) Equation (NP4.2a) and its primed, starred, and primed starred versions, given, respectively, by equations (NP4.2n), (NP4.2p), and (NP4.2g).

(ii) Equation (NP4.2c) and its primed, starred, and primed starred versions given, respectively, by equations (NP4.2i), (NP4.2k), and (NP4.2m).

(iii) Equation (NP4.2b) and its primed, starred and primed version, equation (NP4.2j). The star operation leaves both of these equations invariant.

(iv), (v) The sum and the difference of (NP4.2h) and (NP4.2q). Both remain invariant under both the star and the prime operations.

(vi) Equation (NP4.2d) and its primed (or starred) version (NP4.2o).

(vii) Equation (NP4.2e) and its primed version (NP4.2r). The star operation leaves both of these equations invariant.

(viii) Equation (NP4.2f) and its starred version (NP4.2l). The prime operation leaves both of these equations invariant.

It must be emphasized that in addition to these equations, we must also consider their conjugates.

Let us interject a word about our way of labeling equations in this paper. Many of our equations will split naturally into sets such that within each set all equations can be obtained from one key equation via the discrete transformations, just as was the case for the spin-coefficient equations. Labeling this key equation as (a), equations (a′), (a*), (ã), etc., will then refer to, respectively, the prime transform, the star transform, the conjugate, etc., of equation (a). More often than not, for the sake of brevity, we shall write out explicitly only the key equation for such a set.

Let us next look at the Bianchi identities as they are listed in Pirani (1965). There are three sets, and in the order presented there they may be numbered as (P1a), (P1a*), (P2a), (P2a*), (P2a′*), (P2a′), (P1a′*), (P1a′), (P3a), (P3a*), P3a′). The conjugates of the first two sets of equations must be considered as well. As for the third set, we need only add the conjugate of (P3a*). This is so since (P3a) and (P3a′) are invariant under conjugation and $(\mathrm{P3a}'^*) = (\widetilde{\mathrm{P3a}^*})$.

The six independent commutator equations, four of which are given by equations (NP(4.4), may be split into two sets. If we label the first two of these equations [as they appear in (NP4.4)] as (a) and (b), then the other two are ($\tilde{\mathrm{b}}'$) and (a*); the remaining two, not listed explicitly in (NP4.4), are ($\tilde{\mathrm{b}}$) and (b′). To verify that there are no further commutator equations to be obtained by applying the discrete transformations to equations (a) and (b), note that equations (a) and (a*) are invariant under conjugation and under the prime operation, that equations (b) and (b′) are invariant under the star operation, and that equations ($\tilde{\mathrm{b}}$) and ($\tilde{\mathrm{b}}'$) get interchanged under the star operation.

Similarly, the Maxwell equations, as listed in equation (NP,A1), may be labeled, in order, as (a), (a′*), (a*), (a′). In addition there are the equations conjugate to these.

3. THE DIAGONAL TRANSFORMATIONS

Of special interest among the $GL(2, C) \otimes \tilde{GL}(2, C)$ transformations of the spin frame are the diagonal elements

$$\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix}\right\}$$

where $ad\tilde{a}\tilde{d} \neq 0$, since they include not only rotations in the m–$\tilde{m}$ plane and boosts in the l–n plane, but also basic conformal rescalings and pure spin transformations. Whereas the GHP formalism deals only with the first two of these four transformations, the formalism to be developed here is covariant under all diagonal transformations. Our aim is to rewrite the NP equations in such a way that each term is a properly weighted quantity not just under boost-rotations, but under all diagonal transformations.

For such diagonal elements the two conformal factors are given by

$$\theta=(ad)^{-1} \qquad \text{and} \qquad \tilde{\theta}=(\tilde{a}\tilde{d})^{-1} \tag{1}$$

Equivalently, we can deal with Ω and ω defined, respectively, by

$$\Omega=(\theta\tilde{\theta})^{1/2}, \qquad \omega=(\tilde{\theta}\theta^{-1})^{1/4} \tag{2}$$

According to Appendix A, the components Ψ_i, $\tilde{\Psi}_i$ $(i=0,\ldots,4)$ of the Weyl tensor as well as the spin coefficients κ, $\rho-\tilde{\rho}$, and the transforms of these under the discrete transformations, i.e.,

$$\kappa,\ \sigma,\ \lambda,\ \nu,\ \tilde{\kappa},\ \tilde{\sigma},\ \tilde{\lambda},\ \tilde{\nu},\ \rho-\tilde{\rho},\ \mu-\tilde{\mu},\ \tau+\tilde{\pi},\ \tilde{\tau}+\pi$$

transform as properly weighted quantities under diagonal transformations. The remaining spin coefficients and the components of the Ricci tensor do not. Instead we have the following sets of transformation laws. For the spin coefficients

$$\hat{\rho}+\hat{\tilde{\rho}}=a\tilde{a}[\rho+\tilde{\rho}-2D\ln\Omega] \tag{3a}$$

$$\hat{\mu}+\hat{\tilde{\mu}}=d\tilde{d}[\mu+\tilde{\mu}+2\Delta\ln\Omega] \tag{3a'}$$

$$\hat{\tau}-\hat{\tilde{\pi}}=a\tilde{d}[\tau-\tilde{\pi}-2\delta\ln\Omega] \tag{3a*}$$

$$\hat{\pi}-\hat{\tilde{\tau}}=d\tilde{a}[\pi-\tilde{\tau}+2\tilde{\delta}\ln\Omega] \tag{3a'*}$$

and

$$\hat{\varepsilon}=a\tilde{a}[\varepsilon+D\ln\omega/d] \tag{4a}$$

$$\hat{\gamma}=d\tilde{d}[\gamma-\Delta\ln\omega/a] \tag{4a'}$$

$$\hat{\beta}=a\tilde{d}[\beta+\delta\ln\omega/d] \tag{4a*}$$

$$\hat{\alpha}=d\tilde{a}[\alpha-\tilde{\delta}\ln\omega/a] \tag{4a'*}$$

as well as the conjugates of equations (4). Conjugation of equations (3) leads to no further equations; (3a) and (3a′) are left invariant and (3a*) and (3a′*) are interchanged. Again it should be noted that the last three of equations (3), for instance, can be obtained from the first by means of the transformation laws of Appendix A. In the following we shall usually write down explicitly only one equation of such a set.

The Ricci tensor transformation laws (Ludwig, 1978) split into the following sets:

$$\hat{\Phi}_{00} = a^2\tilde{a}^2[\Phi_{00} - D^2 \ln\Omega + (D\ln\Omega)^2 + (\varepsilon+\tilde{\varepsilon})D\ln\Omega \\ - \tilde{\kappa}\delta\ln\Omega - \kappa\tilde{\delta}\ln\Omega] \tag{5a}$$

plus (5a′), (5a*), (5a′*),

$$\hat{\Phi}_{01} = a^2\tilde{a}\tilde{d}[\Phi_{01} - D\delta\ln\Omega + (D\ln\Omega)\delta\ln\Omega + \tilde{\pi}D\ln\Omega \\ + (\varepsilon - \tilde{\varepsilon})\delta\ln\Omega - \kappa\Delta\ln\Omega] \tag{6a}$$

plus (6a′), (6ã), (6ã′),

$$\Phi_{11} = \tfrac{1}{2}ad\tilde{a}\tilde{d}[2\Phi_{11} - D\Delta\ln\Omega + (D\ln\Omega)\Delta\ln\Omega - \tilde{\delta}\delta\ln\Omega \\ + (\tilde{\delta}\ln\Omega)\delta\ln\Omega + \tilde{\mu}D\ln\Omega + (\pi + \alpha - \tilde{\beta})\delta\ln\Omega + \tilde{\pi}\tilde{\delta}\ln\Omega \\ - (\rho + \varepsilon + \tilde{\varepsilon})\Delta\ln\Omega] \tag{7}$$

$$\hat{\Lambda} = \Omega^{-2}\{\Lambda + (D\ln\Omega)\Delta\ln\Omega - (\delta\ln\Omega)\tilde{\delta}\ln\Omega \\ + \tfrac{1}{2}[-\tilde{\delta}\delta\ln\Omega + (\tilde{\delta}\ln\Omega)\delta\ln\Omega + D\Delta\ln\Omega - (D\ln\Omega)\Delta\ln\Omega \\ + \tilde{\mu}D\ln\Omega + (\alpha - \pi - \tilde{\beta})\delta\ln\Omega - \tilde{\pi}\tilde{\delta}\ln\Omega + (\varepsilon + \tilde{\varepsilon} - \rho)\Delta\ln\Omega]\} \tag{8}$$

That the remaining discrete operations yield no further equations is easily verified. Conjugation leaves equations (5a), (5a′), (7), and (8) invariant and interchanges (5a*) and (5a′*). Equation (6a) and its star transform are the same, as seen with the aid of a commutator equation applied to Ω. Similarly, the star operation applied to equations (6a′), (6ã), (6ã′), (7), and (8) or the prime operation applied to equations (7), and (8) yields nothing new.

For arbitrary w_0, $\tilde{w}_0$, let us define

$$A = \Omega^{-w_0}a\omega^{-1}, \qquad \tilde{A} = \Omega^{-\tilde{w}_0}\tilde{a}\omega$$

A general diagonal element can then be decomposed into a product of (from right to left) a boost-rotation, a conformal rescaling, and a pure spin transformation as follows:

$$\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix}\right\} \\ = \{\omega I, \omega^{-1} I\}\left\{\begin{pmatrix} \Omega^{w_0} & 0 \\ 0 & \Omega^{-1-w_0} \end{pmatrix}, \begin{pmatrix} \Omega^{\tilde{w}_0} & 0 \\ 0 & \Omega^{-1-\tilde{w}_0} \end{pmatrix}\right\} \\ \times \left\{\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A}^{-1} \end{pmatrix}\right\}$$

The diagonal transformation determines the pure spin transformation essentially uniquely, up to a factor of $\{-I, -I\}$, but the conformal rescaling depends on the choice of w_0 and $\tilde{w}_0$.

We shall be interested in scalar functions η that transform properly under a general diagonal transformation as

$$\hat{\eta} = a^r \tilde{a}^s d^t \tilde{d}^u \eta \tag{9}$$

The set of numbers (r, s, t, u) in equation (9) are the *weights* of η. An equivalent set of weights (p, q, m, n) may be defined by

$$p = r - t, \qquad q = s - u, \qquad m = r + t, \qquad n = s + u \tag{10}$$

Both sets of weights are given in Appendix B for various quantities of interest.

In terms of Ω, ω, A, $\tilde{A}$ and the weights p, q, m, n, equation (9) can be expressed as

$$\hat{\eta} = A^p \tilde{A}^q \Omega^{(w_0+\frac{1}{2})p + (\tilde{w}_0+\frac{1}{2})q - \frac{1}{2}(m+n)} \omega^{m-n} \eta \tag{11}$$

If we consider only boost-rotations (for which $\Omega = \omega = 1$), we see immediately from equation (11) that p and q are the GHP weights, i.e., $\frac{1}{2}(p-q)$ is the *spin weight* and $\frac{1}{2}(p+q)$ the *boost weight.* If we consider only a pure spin transformation (for which $A = \tilde{A} = \Omega = 1$), then equation (11) becomes

$$\hat{\eta} = \omega^{m-n} \eta$$

Hence $m - n$, i.e., $r + t - s - u$, is the *pure spin weight,* which must vanish for tensor-related quantities such as the spin coefficients.

On the other hand, the power of Ω in equation (11) depends on the choice of w_0 and $\tilde{w}_0$. It seems natural to define the *conformal weight* of η to be $-\frac{1}{2}(m+n)$. Contrary to standard usage, the power of Ω in equation (11) is not called the conformal weight of η unless the choice $w_0 = \tilde{w}_0 = -\frac{1}{2}$ is made, i.e., unless the conformal rescaling is taken to be $\{\Omega^{-1/2} I, \Omega^{-1/2} I\}$.

4. DIFFERENTIAL OPERATORS

In equation (5.6.33) of Penrose and Rindler (1984), they define operators Þ_c, $\text{Þ}'_c$, $\eth_c$, $\eth'_c$ which are conformally covariant generalizations of the GHP operators. Since, according to equation (11) the power w of Ω is given by

$$w = p(w_0 + \tfrac{1}{2}) + q(\tilde{w}_0 + \tfrac{1}{2}) - \tfrac{1}{2}(m+n)$$

it is easily verified that in terms of the GHP operators they are given by

$$\text{Þ}_c = \text{Þ} - \rho(r+s) \tag{12a}$$

$$Þ'_c = Þ' + \mu(t+u) \tag{12a'}$$

$$ð_c = ð - \tau(r+u) \tag{12a*}$$

$$ð'_c = ð' + \pi(s+t) \tag{12a'*}$$

These equations confirm the claim made that the new operators are independent of the choice of w_0 and $\tilde{w}_0$. That this is so should come as no surprise, since two choices of w_0 and $\tilde{w}_0$ are related by a boost-rotation and the operators transform as properly weighted quantities under boost-rotations.

As also mentioned in Penrose and Rindler (1984), conjugation does not leave $Þ_c$ and $Þ'_c$ invariant and does not interchange $ð_c$ and $ð'_c$. However, an equivalent set of operators with the desired properties can easily be found. They are the operators D_c, Δ_c, δ_c, $\tilde{\delta}_c$ defined by

$$D_c = D + \varepsilon(t-r) + \tilde{\varepsilon}(u-s) - \tfrac{1}{2}(r+s)(\rho+\tilde{\rho}) \tag{13a}$$

$$\Delta_c = \Delta + \gamma(t-r) + \tilde{\gamma}(u-s) + \tfrac{1}{2}(t+u)(\mu+\tilde{\mu}) \tag{13a'}$$

$$\delta_c = \delta + \beta(t-r) + \tilde{\alpha}(u-s) - \tfrac{1}{2}(r+u)(\tau-\tilde{\pi}) \tag{13a*}$$

$$\tilde{\delta}_c = \tilde{\delta} + \alpha(t-r) + \tilde{\beta}(u-s) + \tfrac{1}{2}(s+t)(\pi-\tilde{\tau}) \tag{13a'*}$$

These differential operators are properly weighted quantities, with weights as shown in Appendix B, when acting on tensor-related (scalar) quantities or when the diagonal transformation does not contain a pure spin transformation component ($\omega = 1, \tilde{\theta} = \theta$). Under these circumstances all four operators have a conformal weight of -1. However, under a general diagonal transformation, the transformation law, for any scalar function η, is given by

$$\hat{D}_c\hat{\eta} = a^{r+1}\tilde{a}^{s+1}d^t\tilde{d}^u[D_c\eta + (r+t-s-u)\eta D \ln \omega] \tag{14}$$

and its primed, starred, and primed starred versions. Note that $r+t-s-u$ is the pure spin weight $m-n$ of η.

To conclude this section, we rewrite the Maxwell equations in terms of our new operators. They are

$$D_c\phi_1 - \tilde{\delta}_c\phi_0 = \tfrac{1}{2}(\pi+\tilde{\tau})\phi_0 + (\rho-\tilde{\rho})\phi_1 - \kappa\phi_2 \tag{15a}$$

along with equations (15a′), (15a*), (15a′*), and their conjugates. Although the first four of these equations are slightly more involved, the conjugate equations are less involved than when written in terms of the operators $Þ_c$, $Þ'_c$, $ð_c$, $ð'_c$. Keeping track of conjugate equations is, in general, easier with our choice of operators.

5. COMMUTATORS

The commutators for the differential operators D_c, Δ_c, δ_c, $\tilde{\delta}_c$ are worked out in a straightforward, if lengthy, manner with the help of the NP commutators, equations (NP4.4). They are given by

$$0=[\delta_c D_c - D_c\delta_c]\eta+\tfrac{1}{2}(\tau+\tilde{\pi})D_c\eta-\tfrac{1}{2}(\rho-\tilde{\rho})\delta_c\eta+\sigma\tilde{\delta}_c\eta-\kappa\Delta_c\eta$$
$$+\tfrac{1}{2}\eta[p\mathfrak{L}_1-9\widetilde{\mathfrak{L}_2}-\tfrac{1}{2}(m+n)\mathfrak{A}] \tag{16a}$$

$$0=[\tilde{\delta}_c D_c - D_c\tilde{\delta}_c]\eta+\tfrac{1}{2}(\tilde{\tau}+\pi)D_c\eta+\tfrac{1}{2}(\rho-\tilde{\rho})\tilde{\delta}_c\eta+\tilde{\sigma}\delta_c\eta-\tilde{\kappa}\Delta_c\eta$$
$$+\tfrac{1}{2}\eta[-p\mathfrak{L}_2+q\widetilde{\mathfrak{L}_1}-\tfrac{1}{2}(m+n)\tilde{\mathfrak{A}}] \tag{16ã}$$

$$0=[\tilde{\delta}_c\Delta_c-\Delta_c\tilde{\delta}_c]\eta-\tfrac{1}{2}(\pi+\tilde{\tau})\Delta_c\eta+\tfrac{1}{2}(\mu-\tilde{\mu})\tilde{\delta}_c\eta-\lambda\delta_c\eta+\nu D_c\eta$$
$$+\tfrac{1}{2}\eta[-p\mathfrak{L}_1'+q\widetilde{\mathfrak{L}_2'}-\tfrac{1}{2}(m+n)\mathfrak{A}'] \tag{16a'}$$

$$0=[\delta_c\Delta_c-\Delta_c\delta_c]\eta-\tfrac{1}{2}(\tau+\tilde{\pi})\Delta_c\eta-\tfrac{1}{2}(\mu-\tilde{\mu})\delta_c\eta-\tilde{\lambda}\tilde{\delta}_c\eta+\tilde{\nu}D_c\eta$$
$$+\tfrac{1}{2}\eta[p\mathfrak{L}_2'-q\widetilde{\mathfrak{L}_1'}-\tfrac{1}{2}(m+n)\widetilde{\mathfrak{A}'}] \tag{16ã'}$$

$$0=[\Delta_c D_c-D_c\Delta_c]\eta+(\tau+\tilde{\pi})\tilde{\delta}_c\eta+(\pi+\tilde{\tau})\delta_c\eta$$
$$+\tfrac{1}{2}\eta[p\mathfrak{L}_3+q\widetilde{\mathfrak{L}_3}+\tfrac{1}{2}(m+n)\mathfrak{B}] \tag{17a}$$

$$0=[\tilde{\delta}_c\delta_c-\delta_c\tilde{\delta}_c]\eta+(\rho-\tilde{\rho})\Delta_c\eta+(\mu-\tilde{\mu})D_c\eta$$
$$+\tfrac{1}{2}\eta[p\mathfrak{L}_3^*-q\widetilde{\mathfrak{L}_3^*}+\tfrac{1}{2}(m+n)\mathfrak{B}^*] \tag{17a*}$$

That the remaining discrete transformations yield no further equations is easily verified, as in Section 2. The quantities $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$, $\mathfrak{L}_1'$, $\mathfrak{L}_2'$, $\mathfrak{L}_3^*$, $\mathfrak{A}$, $\mathfrak{A}'$, their conjugates, and $\mathfrak{B}$ and $\mathfrak{B}^*$ are independent, properly weighted quantities with weights as shown in Appendix B. Their transformation properties under the remaining discrete transformations can be found in Appendix A; for example, $\mathfrak{A}^*=-\mathfrak{A}$. In terms of NP variables, these quantities are given by

$$\mathfrak{A}=D(\tau-\tilde{\pi})-\delta(\rho+\tilde{\rho})+(\rho+\tilde{\rho})(\tilde{\alpha}+\beta-\tilde{\pi})+(\tilde{\pi}-\tau)(\tilde{\rho}+\varepsilon-\tilde{\varepsilon})$$
$$-\kappa(\mu+\tilde{\mu})+\sigma(\pi-\tilde{\tau})$$

$$\mathfrak{B}=D(\mu+\tilde{\mu})+\Delta(\rho+\tilde{\rho})-(\rho+\tilde{\rho})(\gamma+\tilde{\gamma})$$
$$+(\mu+\tilde{\mu})(\varepsilon+\tilde{\varepsilon})+(\tau+\tilde{\pi})(\tilde{\tau}-\pi)+(\tau-\tilde{\pi})(\tilde{\tau}+\pi)$$

$$\mathfrak{L}_1=\tfrac{1}{2}\delta(\rho+\tilde{\rho})+\tfrac{1}{2}D(\tilde{\pi}-\tau)-2(D\beta-\delta\varepsilon)-\tfrac{1}{2}\kappa(4\gamma+\mu+\tilde{\mu})$$
$$+\tfrac{1}{2}\sigma(4\alpha+\pi-\tilde{\tau})+\tfrac{1}{2}\rho(\tilde{\pi}-\beta-\tilde{\alpha})+\tfrac{1}{2}\tilde{\rho}(\tau+3\beta-\tilde{\alpha})$$
$$+\varepsilon(\tfrac{3}{2}\tilde{\pi}+\tfrac{1}{2}\tau-2\tilde{\alpha})+\tilde{\varepsilon}(\tfrac{1}{2}\tilde{\pi}-\tfrac{1}{2}\tau-2\beta)$$

$$\mathfrak{L}_2=2(D\alpha-\tilde{\delta}\varepsilon)+\tfrac{1}{2}D(\pi-\tilde{\tau})-\tfrac{1}{2}\tilde{\delta}(\rho+\tilde{\rho})+\tfrac{1}{2}\tilde{\kappa}(\mu+\tilde{\mu}+4\gamma)$$
$$+\tfrac{1}{2}\tilde{\sigma}(\tilde{\pi}-\tau-4\beta)+\tfrac{1}{2}\rho(\tilde{\beta}-3\alpha-2\pi+\tilde{\tau})+\tfrac{1}{2}\tilde{\rho}(\alpha+\tilde{\beta}-\pi)$$
$$+\varepsilon(4\alpha+2\tilde{\beta}-\tfrac{3}{2}\pi-\tfrac{1}{2}\tilde{\tau})+\tilde{\varepsilon}\left(-2\alpha-\frac{\pi}{2}+\tfrac{1}{2}\tilde{\tau}\right)$$

$$\mathfrak{L}_3=\tfrac{1}{2}\Delta(4\varepsilon+\rho+\tilde{\rho})-\tfrac{1}{2}D(4\gamma+\mu+\tilde{\mu})-\tfrac{1}{2}(\gamma+\tilde{\gamma})(4\varepsilon+\rho+\tilde{\rho})$$
$$-\tfrac{1}{2}(\varepsilon+\tilde{\varepsilon})(4\gamma+\mu+\tilde{\mu})+(\tau+\tilde{\pi})(2\alpha+\pi)+(\tilde{\tau}+\pi)(2\beta-\tilde{\pi})$$

along with the primed, starred, primed starred, and conjugate versions thereof, as appropriate.

That these quantities "transform properly" is evident from the commutator equations that define them. Alternatively, this may be seen by working out NP commutators of $\ln \Omega^2$ as well as of $\ln(ad^{-1})$ using equations (3) and (4). For example, applying δ and D, respectively, to (3a) and (3a*) leads to the transformation law for the commutator quantity $\mathfrak{A}$.

6. RICCI TENSOR EXPRESSIONS

The NP Ricci tensor components do not transform properly under all diagonal transformations. Indeed, their transformation laws are given by equations (5)–(8). If, in these equations, we substitute for the derivatives of $\ln \Omega$ the expressions that are readily obtained from equations (3), we can easily show that the following combinations are properly weighted quantities:

$$\Phi^c_{00} = \Phi_{00} + \tfrac{1}{2}[-D(\rho+\tilde{\rho}) + (\rho+\tilde{\rho})(\varepsilon+\tilde{\varepsilon}) + \tfrac{1}{2}(\rho+\tilde{\rho})^2 + \kappa(\pi-\tilde{\tau}) + \tilde{\kappa}(\tilde{\pi}-\tau)] \tag{18a}$$

along with Φ^c_{22}, Φ^c_{02}, Φ^c_{20} defined by equations (18a′), (18a*), (18a′*); and

$$\Phi^c_{01} = \Phi_{01} + \tfrac{1}{2}[D(\tilde{\pi}-\tau) + (\varepsilon-\tilde{\varepsilon})(\tau-\tilde{\pi}) + \tfrac{1}{2}(\tau+\tilde{\pi})(\rho+\tilde{\rho}) + \kappa(\mu+\tilde{\mu})] \tag{19a}$$

along with $\Phi^c_{21}, \Phi^c_{10}, \Phi^c_{12}$ defined by equations (19a′), (19ã), (19ã′); and

$$\begin{aligned}\Phi^c_{11} = \Phi_{11} + \tfrac{1}{4}[&D\tilde{\mu} - \Delta\rho + \delta\pi + \tilde{\delta}\tilde{\pi} + \tilde{\mu}(\varepsilon+\tilde{\varepsilon}) + \tilde{\pi}(\tilde{\beta}-\alpha)\\ &+ \rho(\gamma+\tilde{\gamma}) + \pi(\beta-\tilde{\alpha}) - \tfrac{1}{2}(\tau-\tilde{\pi})(\tilde{\tau}-\pi) + \tfrac{1}{2}(\rho+\tilde{\rho})(\mu+\tilde{\mu})]\end{aligned} \tag{20}$$

$$\begin{aligned}\Lambda^c = \Lambda + \tfrac{1}{4}[&\Delta\tilde{\rho} - D\mu + \delta\pi + \tilde{\delta}\tilde{\pi} - \mu(\varepsilon+\tilde{\varepsilon}) + \pi(\beta-\tilde{\alpha})\\ &+ \tilde{\pi}(\tilde{\beta}-\alpha) - \tilde{\rho}(\gamma+\tilde{\gamma}) + \tfrac{1}{2}(\rho+\tilde{\rho})(\mu+\tilde{\mu}) + \tfrac{1}{2}(\tau-\tilde{\pi})(\tilde{\tau}-\pi).\end{aligned} \tag{21}$$

Conjugation applied to equations (18) yields nothing new. The star operation applied to equations (19) yields expressions related to the ones of (19) by

$$\Phi^{*c}_{01} - \Phi^c_{01} = \tfrac{1}{2}\mathfrak{A} \tag{22a}$$

along with (22a′), (22ã), (22ã′). Similarly, applying the discrete transformations to (20) and (21) yields expressions related to the previous ones by the equations

$$\begin{aligned}&\Phi'^c_{11} = \Phi^{*c}_{11} = \Phi^c_{11} + \tfrac{1}{4}(\mathfrak{B}^* - \mathfrak{B}), \qquad \tilde{\Phi}^c_{11} = \Phi^c_{11} + \tfrac{1}{4}\mathfrak{B}^*\\ &\Lambda^{*c} = -\Lambda^c, \qquad \tilde{\Lambda}^c = \Lambda^c + \tfrac{1}{4}\mathfrak{B}^*, \qquad \Lambda'^c = \Lambda^c + \tfrac{1}{4}(\mathfrak{B}^* - \mathfrak{B})\end{aligned} \tag{23}$$

along with their transforms under the remaining discrete operations.

Some of the Ricci tensor expressions just defined do not transform under discrete transformations as simply as the corresponding components of the Ricci tensor. Although it would be easy to define such quantities, we chose not to do so, since the Bianchi identities would be more cumbersome in terms of them.

7. THE NP EQUATIONS

The seven spin-coefficient equations in the sets (v)-(viii) of Section 2 can now be written as follows:

$$\text{(NP4.2q-h)} \quad \mathfrak{B}^* - \mathfrak{B} = D_c(\mu - \tilde{\mu}) + \Delta_c(\rho - \tilde{\rho}) - \delta_c(\pi + \tilde{\tau}) - \tilde{\delta}_c(\tau + \tilde{\pi}) \tag{24}$$

$$\text{(NP4.2d)} \quad \mathfrak{L}_2 = \tfrac{1}{2}\tilde{\mathfrak{A}} + \tfrac{1}{2}(\rho - \tilde{\rho})(\pi + \tilde{\tau}) - 2\kappa\lambda + 2\Phi^c_{10} \tag{25a}$$

$$\text{(NP4.2e)} \quad \mathfrak{L}_1 = -\tfrac{1}{2}\mathfrak{A} + \kappa(\mu - \tilde{\mu}) - \sigma(\pi + \tilde{\tau}) - 2\Psi_1 \tag{26a}$$

$$\text{(NP4.2f)} \quad \mathfrak{L}_3 = 2(\Lambda^c - \Phi^c_{11} - \Psi_2 + \kappa\nu) - \tfrac{1}{2}(\tau + \tilde{\pi})(\tilde{\tau} + \pi) \tag{27a}$$

along with equations (25a′), (26a′), (27a*), and their conjugates. From equation (24) and its conjugate we deduce that

$$\mathfrak{B} = \delta_c(\tilde{\tau} + \pi) + \tilde{\delta}_c(\tau + \tilde{\pi}) \tag{28a}$$

$$\mathfrak{B}^* = D_c(\mu - \tilde{\mu}) + \Delta_c(\rho - \tilde{\rho}) \tag{28a*}$$

The remaining eleven equations are

$$\text{(NP4.2a)} \quad \tfrac{1}{2}D_c(\rho - \tilde{\rho}) - \tilde{\delta}_c\kappa = \Phi^c_{00} + \sigma\tilde{\sigma} + \tfrac{1}{2}\kappa(\tilde{\tau} + \pi) - \tfrac{1}{2}\tilde{\kappa}(\tau + \tilde{\pi}) + \tfrac{1}{4}(\rho - \tilde{\rho})^2 \tag{29a}$$

$$\text{(NP4.2c)} \quad \tfrac{1}{2}D_c(\tau + \tilde{\pi}) - \Delta_c\kappa = \Phi^c_{01} + \Psi_1 + \sigma(\pi + \tilde{\tau}) + \tfrac{1}{2}(\tau + \tilde{\pi})(\rho - \tilde{\rho}) \tag{30a}$$

$$\text{(NP4.2b)} \quad D_c\sigma - \delta_c\kappa = \Psi_0 \tag{31a}$$

$$\text{(NP4.2q+h)} \quad \tilde{\delta}_c(\tau + \tilde{\pi}) - \Delta_c(\rho - \tilde{\rho}) = 4\Lambda^c + 2\Psi_2 + 2\lambda\sigma - 2\kappa\nu + \tfrac{1}{2}(\tau + \tilde{\pi})(\tilde{\tau} + \pi) - \tfrac{1}{2}(\rho - \tilde{\rho})(\mu - \tilde{\mu}) \tag{32}$$

along with (29a′), (29a*), (29a′*), (30a′), (30a*), (30a′*), (31a′). In addition to these 18 spin-coefficient equations, we also have their conjugates. In a manner analogous to what is done in the GHP formalism, we eliminate the commutator quantities $\mathfrak{A}, \mathfrak{B}$, etc., from the formalism by substituting equations (22) and (25)-(28) into the commutator equations (16) and (17).

It remains to show the form the Bianchi identities assume in the extended formalism. It suffices to work out one equation in each of the three sets and obtain the remaining equations via the discrete symmetries. A somewhat lengthy but straightforward calculation shows that the identities

$$\tilde{\delta}_c[(31\text{a})] + D_c[(30\text{a}^*)] - \delta_c[(29\text{a})] = 0$$

$$\Delta_c[(29\text{a})] - D_c[(32)] - \tilde{\delta}_c[(30\text{a})] = 0$$

lead, respectively, to equations (P1) and a combination of equations (P2) and (P3), rewritten in terms of properly weighted quantities. A second, independent, combination of equations (P2) and (P3) is obtained after another somewhat lengthy calculation from the identity

$$\tilde{\delta}_c[(16\text{a})] - \delta_c[(16\tilde{\text{a}})] + D_c[(17\text{a}^*)] = 0$$

All this labor results in the following form for the Bianchi identities.

$$\begin{aligned}
\text{P1:}\quad & \tilde{\delta}_c\Psi_0 + D_c(\Phi^{*c}_{01} - \Psi_1) - \delta_c\Phi^c_{00} \\
&= -\tfrac{1}{2}(\tilde{\tau} + \pi)\Psi_0 - \tilde{\kappa}\Phi^c_{02} + \tfrac{1}{2}(\tau + \tilde{\pi})\Phi^c_{00} + \sigma[\Phi^c_{10} + \Phi^{*c}_{12}] \\
&\quad - \tfrac{1}{2}(\rho - \tilde{\rho})[\Phi^c_{01} + \Phi^{*c}_{01} + 4\Psi_1] + \kappa[3\Psi_2 - 2\Phi^{*c}_{11}] \qquad (33\text{a})
\end{aligned}$$

$$\begin{aligned}
\text{P2:}\quad & D_c[2\Phi^{*c}_{11} - 3\Psi_2 + \tfrac{1}{2}\mathfrak{B}^*] + \tilde{\delta}_c[3\Psi_1 + \Phi^c_{01}] - 2\delta_c\Phi^c_{10} - \Delta_c\Phi^c_{00} \\
&= 3\lambda\Psi_0 + \tfrac{1}{2}(\rho - \tilde{\rho})[-9\Psi_2 - 6\Phi^c_{11} - \mathfrak{B}^*] + \tfrac{1}{2}(\pi + \tilde{\tau})[4\Phi^{*c}_{01} - 6\Psi_1] \\
&\quad + \kappa[6\Psi_3 - 3\Phi^c_{21} + \Phi^{*c}_{21}] - 2\tilde{\kappa}\Phi^{*c}_{10} - \tfrac{3}{2}(\mu - \tilde{\mu})\Phi^c_{00} + 2\sigma\Phi^c_{20} \\
&\quad - \tilde{\sigma}\Phi^c_{02} + \tfrac{1}{2}(\tau + \tilde{\pi})[3\Phi^c_{10} + \Phi^{*c}_{12}] \qquad (34\text{a})
\end{aligned}$$

$$\begin{aligned}
\text{P3:}\quad & D_c(3\Lambda^c + \tfrac{1}{4}\mathfrak{B}^* + \Phi^{*c}_{11}) + \Delta_c\Phi^c_{00} - \delta_c\Phi^c_{10} - \tilde{\delta}_c\Phi^c_{01} \\
&= \sigma\Phi^c_{20} + \tilde{\sigma}\Phi^c_{02} - \kappa\Phi^{*c}_{21} - \tilde{\kappa}\Phi^{*c}_{10} - \tfrac{1}{2}(\tilde{\tau} + \pi)\Phi^{*c}_{01} \\
&\quad - \tfrac{1}{2}(\tau + \tilde{\pi})\Phi^{*c}_{12} - \tfrac{1}{4}(\rho - \tilde{\rho})\mathfrak{B}^* \qquad (35\text{a})
\end{aligned}$$

along with (33a′), (33a*), (33a′*), (34a′), (34a*), (34a′*), (35a′), (35a*). The commutator quantities $\mathfrak{B}$ and $\mathfrak{B}^*$ that occur in these equations can be eliminated with the help of equations (23).

8. CONCLUSION

We have presented an extension of the GHP formalism that is covariant not just under boosts and rotations, but under all diagonal transformations of the spin frame, including conformal rescalings. All quantities involved are properly weighted, obeying the transformation law given by equation (9) with the corresponding weights being found in Table I of Appendix B. In particular, the differential operators D_c, Δ_c, δ_c, $\tilde{\delta}_c$, defined by equations

(13) as generalizations of the corresponding NP or GHP operators, have this property. Although we have presented the complex formalism, the specialization to the case of real general relativity is obvious and straightforward.

Full use is made of the formalism's symmetry under all discrete operations, including not only conjugation and the prime operation, but also the (modified) Sachs star operations. As a result, all equations of interest occur in sets of maximally eight equations such that each member of the set can be obtained from any other by applying these discrete operations.

All NP equations are rewritten so that they contain only terms that have the same symmetry (if any) under the discrete operations and that transform as properly weighted quantities, of the same weight, under diagonal transformations. The commutator equations for the four differential operators are given by equations (16) and (17), with the commutator quantities $\mathfrak{A}$, $\mathfrak{B}$, etc., defined by equations (22) and (25)-(28). The spin-coefficient equations and the Bianchi identities are, respectively, equations (29)-(32) and (33)-(35).

The contents of the Einstein equations are a bit more difficult to express in the extended formalism than in either of the NP or GHP formalisms. The properly weighted Ricci tensor expressions must be calculated by substituting the appropriate Ricci tensor components into equations (18)-(21). However, once this is done, the simplification in comparison with the other two formalisms is considerable. This is particularly so when a conformal spacelike surface (and therefore two null directions) are singled out in a natural way as is the case, for example, in considerations of future null infinity.

APPENDIX A. CHANGES UNDER DISCRETE TRANSFORMATIONS

For convenience we summarize here the effect of the discrete transformations on some of the NP variables and other quantities introduced in the course of this work.

A1. The Prime Operation

The prime operation commutes with both the star operation and conjugation. It leaves invariant

$$g^{ab}, g_{ab}, \theta, \tilde{\theta}, \Omega, \omega, m, n, \phi_1, \Psi_2, \Phi_{11}, \Lambda, \mathfrak{L}_3, \tilde{\mathfrak{L}}_3, \mathfrak{L}_3^*, \widetilde{\mathfrak{L}_3^*}$$

It gives rise to the following interchanges.

$$o^A \leftrightarrow \iota^A, \quad \tilde{o}^{\dot{A}} \leftrightarrow \tilde{\iota}^{\dot{A}}, \quad o_A \leftrightarrow -\iota_A, \quad \tilde{o}_{\dot{A}} \leftrightarrow -\tilde{\iota}_{\dot{A}}$$

$$\varepsilon^{AB} \to -\varepsilon^{AB}, \quad \varepsilon_{AB} \to -\varepsilon_{AB}, \quad \varepsilon^{\dot{A}\dot{B}} \to -\varepsilon^{\dot{A}\dot{B}}, \quad \varepsilon_{\dot{A}\dot{B}} \to -\varepsilon_{\dot{A}\dot{B}}$$

$$l^a \leftrightarrow n^a, \quad m^a \leftrightarrow \tilde{m}^a, \quad l_a \leftrightarrow n_a, \quad m_a \leftrightarrow \tilde{m}_a, \quad D \leftrightarrow \Delta, \quad \delta \leftrightarrow \tilde{\delta}$$

$$a \leftrightarrow d, \quad \tilde{a} \leftrightarrow \tilde{d}, \quad r \leftrightarrow t, \quad s \leftrightarrow u, \quad p \to -p, \quad q \to -q, \quad \phi_0 \leftrightarrow \phi_2$$

$$\kappa \leftrightarrow -\nu, \quad \sigma \leftrightarrow -\lambda, \quad \rho \leftrightarrow -\mu, \quad \tau \leftrightarrow -\pi, \quad \varepsilon \leftrightarrow -\gamma, \quad \beta \leftrightarrow -\alpha$$

$$\Psi_0 \leftrightarrow \Psi_4, \quad \Psi_1 \leftrightarrow \Psi_3, \quad \Phi_{00} \leftrightarrow \Phi_{22}, \quad \Phi_{02} \leftrightarrow \Phi_{20}, \quad \Phi_{01} \leftrightarrow \Phi_{21}, \quad \Phi_{10} \leftrightarrow \Phi_{12}$$

$$D_c \leftrightarrow \Delta_c, \quad \delta_c \leftrightarrow \tilde{\delta}_c$$

$$\mathfrak{A} \leftrightarrow \mathfrak{A}', \quad \tilde{\mathfrak{A}} \leftrightarrow \tilde{\mathfrak{A}}', \quad \mathfrak{B} \to -\mathfrak{B}, \quad \mathfrak{B}^* \to -\mathfrak{B}^*$$

$$\mathfrak{L}_1 \leftrightarrow \mathfrak{L}_1', \quad \mathfrak{L}_2 \leftrightarrow \mathfrak{L}_2', \quad \tilde{\mathfrak{L}}_1 \leftrightarrow \tilde{\mathfrak{L}}_1', \quad \tilde{\mathfrak{L}}_2 \leftrightarrow \tilde{\mathfrak{L}}_2'$$

$$\Phi_{00}^c \leftrightarrow \Phi_{22}^c, \quad \Phi_{02}^c \leftrightarrow \Phi_{20}^c, \quad \Phi_{01}^c \leftrightarrow \Phi_{21}^c, \quad \Phi_{10}^c \leftrightarrow \Phi_{12}^c$$

$$\Phi_{11}^c \leftrightarrow \Phi_{11}^c + \tfrac{1}{4}(\mathfrak{B}^* - \mathfrak{B}), \quad \Lambda^c \leftrightarrow \Lambda^c + \tfrac{1}{4}(\mathfrak{B}^* - \mathfrak{B})$$

A2. The Star Operation

The star operation commutes with the prime operation, but not with conjugation. Instead, we have

$$\widetilde{\eta^*} = (\tilde{\eta})'^*$$

for arbitrary η. Left invariant are

$$o^A, \iota^A, o_A, \iota_A, \varepsilon^{AB}, \varepsilon_{AB}, a, d, \theta, \tilde{\theta}, \Omega, \omega, r, t, p, m, n, \Phi_{01}, \Phi_{21}, \Phi_{11}, \tilde{\mathfrak{L}}_2, \widetilde{\mathfrak{L}_2'}$$

We also have the following interchanges:

$$\tilde{o}^{\dot{A}} \leftrightarrow \tilde{\iota}^{\dot{A}}, \quad \tilde{o}_{\dot{A}} \leftrightarrow -\tilde{\iota}_{\dot{A}}, \quad \varepsilon^{\dot{A}\dot{B}} \to -\varepsilon^{\dot{A}\dot{B}}, \quad \varepsilon_{\dot{A}\dot{B}} \to -\varepsilon_{\dot{A}\dot{B}},$$

$$g^{ab} \to -g^{ab}, \quad g_{ab} \to -g_{ab}$$

$$l^a \leftrightarrow m^a, \quad n^a \leftrightarrow \tilde{m}^a, \quad l_a \leftrightarrow -m_a, \quad \tilde{m}_a \leftrightarrow -n_a, \quad D \leftrightarrow \delta, \quad \Delta \leftrightarrow \tilde{\delta}$$

$$\tilde{a} \leftrightarrow \tilde{d}, \quad s \leftrightarrow u, \quad q \to -q$$

$$\phi_i \to -\phi_i \quad (i = 0, 1, 2), \quad \tilde{\phi}_1 \to -\tilde{\phi}_1, \quad \tilde{\phi}_0 \leftrightarrow -\tilde{\phi}_2$$

$$\kappa \leftrightarrow \sigma, \quad \rho \leftrightarrow \tau, \quad \pi \leftrightarrow \mu, \quad \lambda \leftrightarrow \nu, \quad \alpha \leftrightarrow \gamma, \quad \beta \leftrightarrow \varepsilon$$

$$\tilde{\kappa} \leftrightarrow -\tilde{\lambda}, \quad \tilde{\sigma} \leftrightarrow -\tilde{\nu}, \quad \tilde{\rho} \leftrightarrow -\tilde{\pi}, \quad \tilde{\tau} \leftrightarrow -\tilde{\mu}, \quad \tilde{\alpha} \leftrightarrow -\tilde{\varepsilon}, \quad \tilde{\beta} \leftrightarrow -\tilde{\gamma}$$

$$\Psi_i \to -\Psi_i \quad (i = 0, \ldots, 4), \quad \tilde{\Psi}_0 \leftrightarrow -\tilde{\Psi}_4, \quad \tilde{\Psi}_1 \leftrightarrow -\tilde{\Psi}_3, \quad \tilde{\Psi}_2 \to -\tilde{\Psi}_2$$

$$\Lambda \to -\Lambda, \quad \Phi_{00} \leftrightarrow \Phi_{02}, \quad \Phi_{20} \leftrightarrow \Phi_{22}, \quad \Phi_{10} \leftrightarrow \Phi_{12}$$

$$D_c \leftrightarrow \delta_c, \quad \Delta_c \leftrightarrow \tilde{\delta}_c$$

$$\mathfrak{A} \to -\mathfrak{A}, \quad \mathfrak{A}' \to -\mathfrak{A}', \quad \tilde{\mathfrak{A}} \leftrightarrow -\tilde{\mathfrak{A}}', \quad \mathfrak{B} \leftrightarrow \mathfrak{B}^*$$

$$\mathfrak{L}_1 \to -\mathfrak{L}_1, \quad \mathfrak{L}_1' \to -\mathfrak{L}_1', \quad \tilde{\mathfrak{L}}_1 \leftrightarrow -\widetilde{\mathfrak{L}_1'}, \quad \mathfrak{L}_2 \leftrightarrow \mathfrak{L}_2', \quad \mathfrak{L}_3 \leftrightarrow \mathfrak{L}_3^*, \quad \tilde{\mathfrak{L}}_3 \leftrightarrow \widetilde{\mathfrak{L}_3^*}$$

$$\Lambda^c \to -\Lambda^c, \quad \Phi_{00}^c \leftrightarrow \Phi_{02}^c, \quad \Phi_{20}^c \leftrightarrow \Phi_{22}^c, \quad \Phi_{10}^c \leftrightarrow \Phi_{12}^c + \tfrac{1}{2}\tilde{\mathfrak{A}}'$$

$$\Phi_{01}^c \leftrightarrow \Phi_{01}^c + \tfrac{1}{2}\mathfrak{A}, \quad \Phi_{21}^c \leftrightarrow \Phi_{21}^c + \tfrac{1}{2}\mathfrak{A}', \quad \Phi_{11}^c \leftrightarrow \Phi_{11}^c + \tfrac{1}{4}(\mathfrak{B}^* - \mathfrak{B})$$

A3. Conjugation

In general, the transform of a variable η under conjugation, i.e., under a basic reflection (Ludwig, 1987), is denoted by $\tilde{\eta}$. The two variables are independent except in the real case where they are complex conjugates. Conjugation commutes with the prime operation, but not with the star operation. Instead, we have

$$\widetilde{\eta^*} = (\tilde{\eta})'^*$$

The variables

$$g_{ab}, \Omega, D, \Delta, \Lambda, \Phi_{00}, \Phi_{11}, \Phi_{22}, D_c, \Delta_c, \mathfrak{B}, \Phi^c_{00}, \Phi^c_{22}$$

remain invariant under conjugation. These self-conjugate variables are real in the real case. Conjugation, in general, interchanges tilded and untilded variables; it also results in the following less obvious interchanges:

$$\omega \to \omega^{-1}, \quad r \leftrightarrow s, \quad u \leftrightarrow t, \quad p \leftrightarrow q, \quad m \leftrightarrow n$$

$$\Phi_{01} \leftrightarrow \Phi_{10}, \quad \Phi_{02} \leftrightarrow \Phi_{20}, \quad \Phi_{12} \leftrightarrow \Phi_{21}$$

$$\mathfrak{B}^* \to -\mathfrak{B}^*, \quad \Lambda^c \leftrightarrow \Lambda^c + \tfrac{1}{4}\mathfrak{B}^*, \quad \Phi^c_{11} \leftrightarrow \Phi^c_{11} + \tfrac{1}{4}\mathfrak{B}^*$$

$$\Phi^c_{02} \leftrightarrow \Phi^c_{20}, \quad \Phi^c_{01} \leftrightarrow \Phi^c_{10}, \quad \Phi^c_{12} \leftrightarrow \Phi^c_{21}$$

APPENDIX B. WEIGHTS

Properly weighted variables are subject, under diagonal transformations, to the transformation law given by equation (9). The set of weights r, s, t, u and the equivalent set p, q, $-\frac{1}{2}(m+n)$, $m-n$ defined by Eq. (10) are listed in Table I for various quantities of interest. The latter set of weights give, respectively, the GHP weights, the conformal weight, and the pure spin weight. For tensor-related quantities (Ludwig, 1987) the pure spin weight vanishes.

The variables split naturally into sets. In each set the weights of all variables could be obtained from those of the first member with the aid of the transformation properties of these variables under discrete transformations, as given in Appendix A or C. However, for convenience, we list the weights of all variables in each set.

APPENDIX C. THE EXTENDED GHP NOTATION

The NP formalism does not employ a new Greek symbol for the conjugate of a variable η, denoting it by $\tilde{\eta}$ instead. The GHP notation makes full use of the symmetry of the formalism under both conjugation and the prime operation, denoting the transforms of η by $\tilde{\eta}$ and η',

Table I

Variables	r	s	t	u	p	q	$-\frac{1}{2}(m+n)$	$m-n$
o^A	1	0	0	0	1	0	$-\frac{1}{2}$	1
$\tilde{o}^{\dot{A}}$	0	1	0	0	0	1	$-\frac{1}{2}$	-1
ι^A	0	0	1	0	-1	0	$-\frac{1}{2}$	1
$\tilde{\iota}^{\dot{A}}$	0	0	0	1	0	-1	$-\frac{1}{2}$	-1
o_A	0	0	-1	0	1	0	$\frac{1}{2}$	-1
$\tilde{o}_{\dot{A}}$	0	0	0	-1	0	1	$\frac{1}{2}$	1
ι_A	-1	0	0	0	-1	0	$\frac{1}{2}$	-1
$\tilde{\iota}_{\dot{A}}$	0	-1	0	0	0	-1	$\frac{1}{2}$	1
ε^{AB}	1	0	1	0	0	0	-1	2
$\varepsilon^{\dot{A}\dot{B}}$	0	1	0	1	0	0	-1	-2
ε_{AB}	-1	0	-1	0	0	0	1	-2
$\varepsilon_{\dot{A}\dot{B}}$	0	-1	0	-1	0	0	1	2
g_{ab}	-1	-1	-1	-1	0	0	2	0
l_a	0	0	-1	-1	1	1	1	0
n_a	-1	-1	0	0	-1	-1	1	0
m_a	0	-1	-1	0	1	-1	1	0
$\tilde{m}_a$	-1	0	0	-1	-1	1	1	0
$l^a, D_c, \rho-\tilde{\rho}$	1	1	0	0	1	1	-1	0
$n^a, \Delta_c, \mu-\tilde{\mu}$	0	0	1	1	-1	-1	-1	0
$m^a, \delta_c, \tau+\tilde{\pi}$	1	0	0	1	1	-1	-1	0
$\tilde{m}^a, \tilde{\delta}_c, \tilde{\tau}+\pi$	0	1	1	0	-1	1	-1	0
κ	2	1	-1	0	3	1	-1	0
ν	-1	0	2	1	-3	-1	-1	0
σ	2	0	-1	1	3	-1	-1	0
λ	-1	1	2	0	-3	1	-1	0
$\tilde{\kappa}$	1	2	0	-1	1	3	-1	0
$\tilde{\nu}$	0	-1	1	2	-1	-3	-1	0
$\tilde{\sigma}$	0	2	1	-1	-1	3	-1	0
$\tilde{\lambda}$	1	-1	0	2	1	-3	-1	0
Ψ_0	3	1	-1	1	4	0	-2	0
$\tilde{\Psi}_0$	1	3	1	-1	0	4	-2	0
Ψ_4	-1	1	3	1	-4	0	-2	0
$\tilde{\Psi}_4$	1	-1	1	3	0	-4	-2	0
$\phi_0, \Psi_1, \mathfrak{A}, \mathfrak{L}_1, \tilde{\mathfrak{L}}_2, \Phi^c_{01}$	2	1	0	1	2	0	-2	0
$\tilde{\phi}_0, \tilde{\Psi}_1, \tilde{\mathfrak{A}}, \tilde{\mathfrak{L}}_1, \mathfrak{L}_2, \Phi^c_{10}$	1	2	1	0	0	2	-2	0
$\phi_2, \Psi_3, \mathfrak{A}', \mathfrak{L}'_1, \tilde{\mathfrak{L}}'_2, \Phi^c_{21}$	0	1	2	1	-2	0	-2	0
$\tilde{\phi}_2, \tilde{\Psi}_3, \tilde{\mathfrak{A}}', \tilde{\mathfrak{L}}'_1, \mathfrak{L}'_2, \Phi^c_{12}$	1	0	1	2	0	-2	-2	0
Φ^c_{00}	2	2	0	0	2	2	-2	0
Φ^c_{22}	0	0	2	2	-2	-2	-2	0
Φ^c_{02}	2	0	0	2	2	-2	-2	0
Φ^c_{20}	0	2	2	0	-2	2	-2	0
$\phi_1, \tilde{\phi}_1, \Psi_2, \tilde{\Psi}_2, \mathfrak{B}, \mathfrak{B}^*, \mathfrak{L}_3, \mathfrak{L}^*_3, \tilde{\mathfrak{L}}_3, \widetilde{\mathfrak{L}^*_3}, \Phi^c_{11}, \Lambda^c, g^{ab}$	1	1	1	1	0	0	-2	0

Table II

New notation	Usual notation	Symmetries
$a, \tilde{a}, a', \tilde{a}'$	$a, \tilde{a}, d, \tilde{d}$	$a^* = a$
$\theta, \tilde{\theta}$	$\theta, \tilde{\theta}$	$\theta' = \theta^* = \theta$
Ω	Ω	$\Omega' = \Omega^* = \tilde{\Omega} = \Omega$
ω	ω	$\omega' = \omega^* = \tilde{\omega}^{-1} = \omega$
$r, \tilde{r}, r', \tilde{r}'$	r, s, t, u	$r^* = r$
$p, \tilde{p}$	p, q	$-p' = p^* = p$
$m, \tilde{m}$	m, n	$m' = m^* = m$
$o^A, \tilde{o}^{\dot{A}}, o'^A, \tilde{o}'^{\dot{A}}$	$o^A, \tilde{o}^{\dot{A}}, \iota^A, \tilde{\iota}^{\dot{A}}$	$o^{*A} = o^A$
$o_A, \tilde{o}_{\dot{A}}, o'_A, \tilde{o}'_{\dot{A}}$	$o_A, \tilde{o}_{\dot{A}}, -\iota_A, -\tilde{\iota}_{\dot{A}}$	$o^*_A = o_A$
$\varepsilon_{AB}, \varepsilon_{\dot{A}\dot{B}}$	$\varepsilon_{AB}, \varepsilon_{\dot{A}\dot{B}}$	$-\varepsilon'_{AB} = \varepsilon^*_{AB} = \varepsilon_{AB}$
g_{ab}	g_{ab}	$g'_{ab} = -g^*_{ab} = \tilde{g}_{ab} = g_{ab}$
l_a, l'_a, l^*_a, l'^*_a	$l_a, n_a, -m_a, -\tilde{m}_a$	$\tilde{l}_a = l_a$
$l^a, l'^a, l^{*a}, l'^{*a}$	$l^a, n^a, m^a, \tilde{m}^a$	$\tilde{l}^a = l^a$
D, D', D^*, D'^*	$D, \Delta, \delta, \tilde{\delta}$	$\tilde{D} = D$
$\phi_0, \tilde{\phi}_0, \phi'_0, \tilde{\phi}'_0$	$\phi_0, \tilde{\phi}_0, \phi_2, \tilde{\phi}_2$	$\phi^*_0 = -\phi_0$
$\phi_1, \tilde{\phi}_1$	$\phi_1, \tilde{\phi}_1$	$-\phi^*_1 = \phi'_1 = \phi_1$
$\kappa, \kappa', \kappa^*, \kappa'^*, \tilde{\kappa}, \tilde{\kappa}', (\tilde{\kappa})'^*, (\tilde{\kappa})^*$	$\kappa, -\nu, \sigma, -\lambda, \tilde{\kappa}, -\tilde{\nu}, \tilde{\sigma}, -\tilde{\lambda}$	
$\rho, \rho', \rho^*, \rho'^*, \tilde{\rho}, \tilde{\rho}', (\tilde{\rho})'^*, (\tilde{\rho})^*$	$\rho, -\mu, \tau, -\pi, \tilde{\rho}, -\tilde{\mu}, \tilde{\tau}, -\tilde{\pi}$	
$\varepsilon, \varepsilon', \varepsilon^*, \varepsilon'^*, \tilde{\varepsilon}, \tilde{\varepsilon}', (\tilde{\varepsilon})'^*, (\tilde{\varepsilon})^*$	$\varepsilon, -\gamma, \beta, -\alpha, \tilde{\varepsilon}, -\tilde{\gamma}, \tilde{\beta}, -\tilde{\alpha}$	
$\rho - \tilde{\rho}, (\rho - \tilde{\rho})', (\rho - \tilde{\rho})^*,$	$\rho - \tilde{\rho}, -(\mu - \tilde{\mu}),$	
$(\rho - \tilde{\rho})'^*$	$\tau + \tilde{\pi}, -(\tilde{\tau} + \pi)$	$\widetilde{(\rho - \tilde{\rho})} = -(\rho - \tilde{\rho})$
$\Psi_0, \tilde{\Psi}_0, \Psi'_0, \tilde{\Psi}'_0$	$\Psi_0, \tilde{\Psi}_0, \Psi_4, \tilde{\Psi}_4$	$\Psi^*_0 = -\Psi_0$
$\Psi_1, \tilde{\Psi}_1, \Psi'_1, \tilde{\Psi}'_1$	$\Psi_1, \tilde{\Psi}_1, \Psi_3, \tilde{\Psi}_3$	$\Psi^*_1 = -\Psi_1$
$\Psi_2, \tilde{\Psi}_2$	$\Psi_2, \tilde{\Psi}_2$	$-\Psi^*_2 = \Psi'_2 = \Psi_2$
$\Phi_{00}, \Phi'_{00}, \Phi^*_{00}, \Phi'^*_{00}$	$\Phi_{00}, \Phi_{22}, \Phi_{02}, \Phi_{20}$	$\tilde{\Phi}_{00} = \Phi_{00}$
$\Phi_{01}, \tilde{\Phi}_{01}, \Phi'_{01}, \tilde{\Phi}'_{01}$	$\Phi_{01}, \Phi_{10}, \Phi_{21}, \Phi_{12}$	$\Phi^*_{01} = \Phi_{01}$
Φ_{11}	Φ_{11}	$\Phi'_{11} = \Phi^*_{11} = \tilde{\Phi}_{11} = \Phi_{11}$
Λ	Λ	$-\Lambda^* = \Lambda' = \tilde{\Lambda} = \Lambda$
D_c, D'_c, D^*_c, D'^*_c	$D_c, \Delta_c, \delta_c, \tilde{\delta}_c$	$\tilde{D}_c = D_c$
$\mathfrak{A}, \tilde{\mathfrak{A}}, \mathfrak{A}', \tilde{\mathfrak{A}}'$	$\mathfrak{A}, \tilde{\mathfrak{A}}, \mathfrak{A}', \tilde{\mathfrak{A}}'$	$\mathfrak{A}^* = -\mathfrak{A}$
$\mathfrak{B}, \mathfrak{B}^*$	$\mathfrak{B}, \mathfrak{B}^*$	$-\mathfrak{B}' = \tilde{\mathfrak{B}} = \mathfrak{B}$
$\mathfrak{L}_1, \tilde{\mathfrak{L}}_1, \mathfrak{L}'_1, \tilde{\mathfrak{L}}'_1$	$\mathfrak{L}_1, \tilde{\mathfrak{L}}_1, \mathfrak{L}'_1, \tilde{\mathfrak{L}}'_1$	$\mathfrak{L}^*_1 = -\mathfrak{L}_1$
$\mathfrak{L}_2, \tilde{\mathfrak{L}}_2, \mathfrak{L}'_2, \tilde{\mathfrak{L}}'_2$	$\mathfrak{L}_2, \tilde{\mathfrak{L}}_2, \mathfrak{L}'_2, \tilde{\mathfrak{L}}'_2$	$(\tilde{\mathfrak{L}}_2)^* = \widetilde{\mathfrak{L}_2}$
$\mathfrak{L}_3\ \mathfrak{L}^*_3, \widetilde{\mathfrak{L}_3}, \widetilde{\mathfrak{L}^*_3}$	$\mathfrak{L}_3, \mathfrak{L}^*_3, \widetilde{\mathfrak{L}_3}, \widetilde{\mathfrak{L}^*_3}$	$\mathfrak{L}'_3 = \mathfrak{L}_3$
$\Phi^c_{00}, \Phi'^c_{00}, \Phi^{*c}_{00}, \Phi'^{*c}_{00}$	$\Phi^c_{00}, \Phi^c_{22}, \Phi^c_{02}, \Phi^c_{20}$	$\tilde{\Phi}^c_{00} = \Phi^c_{00}$
$\Phi^c_{01}, \tilde{\Phi}^c_{01}, \Phi'^c_{01}, \tilde{\Phi}'^c_{01}$	$\Phi^c_{01}, \Phi^c_{10}, \Phi^c_{21}, \Phi^c_{12}$	$\Phi^{*c}_{01} = \Phi^c_{01} + \frac{1}{2}\mathfrak{A}$
Φ^c_{11}	Φ^c_{11}	$\Phi'^c_{11} = \Phi^c_{11} + \frac{1}{4}(\mathfrak{B}^* - \mathfrak{B})$
		$\Phi^{*c}_{11} = \Phi^c_{11} + \frac{1}{4}(\mathfrak{B}^* - \mathfrak{B})$
		$\tilde{\Phi}^c_{11} = \Phi^c_{11} + \frac{1}{4}\mathfrak{B}^*$
Λ^c	Λ^c	$\Lambda^{*c} = -\Lambda^c$
		$\Lambda'^c = \Lambda^c + \frac{1}{4}(\mathfrak{B}^* - \mathfrak{B})$
		$\tilde{\Lambda}^c = \Lambda^c + \frac{1}{4}\mathfrak{B}^*$

respectively. It seems logical to have the symmetry of the formalism under the star operation also reflected in the notation.

Applying the discrete operations to a quantity η, we obtain, in general, eight different quantities, η, η', η^*, η'^*, $\tilde{\eta}$, $\tilde{\eta}'$, $(\tilde{\eta})'^*$, $(\tilde{\eta})^*$. If η is symmetric under one or more of these operations, this number is reduced by a factor $\frac{1}{2}$ for each symmetry. For example, the discrete operations applied to D yield only the four independent operators D, D', D^*, D'^*, i.e., D, Δ, δ, $\tilde{\delta}$. This is due to the fact that D is self-conjugate, i.e., $\tilde{D}=D$. As our method of labeling equations already suggests, the same reasoning applies to each set of equations. For example, there are only two equations in the last set of commutator equations, equations (17). This is due to two symmetries, namely the invariance of equation (17a) under both conjugation and the prime operation.

In Table II we list all variables of the formalism (and others) using both notations. The variables split naturally into sets of the kind just described, each set having one, two, four, or eight members, depending on symmetry. These symmetries, if any, are also exhibited. The content of Appendix A can be deduced from this table.

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